

2020 John O'Bryan Mathematics Competition :: 5-Person Team Test

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper.

4. Let $a \neq -1$.

(a) Calculate $\frac{a^5 + 1}{a + 1}$.

(b) Assume $a + 1$ is divisible by 5. Show that $a^4 - a^3 + a^2 - a + 1$ is divisible by 5.

(c) Let $k \geq 1$ and assume $4^{5^{k-1}} + 1$ is divisible by 5^k . Show that $4^{5^k} + 1$ is divisible by 5^{k+1} .

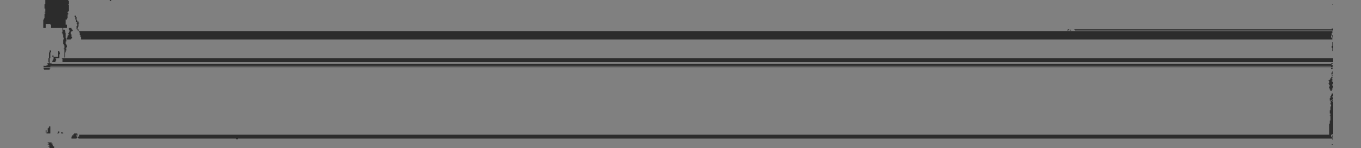
5. In discrete time steps (i.e. $t = 0, 1, 2, 3, 4, \dots$), a contagion spreads around vertices in a graph where infected vertices are solid and healthy vertices are open. If an infected vertex has only one healthy neighbor, that healthy neighbor becomes (and stays) infected at the next time step. Starting (below



(a) Determine the maximum rook social distance on the chess boards below.



(b) How many ways can 3 rooks be placed on a 3×3 chess board so that the squares they are



Freshman-Sophomore Individual Test

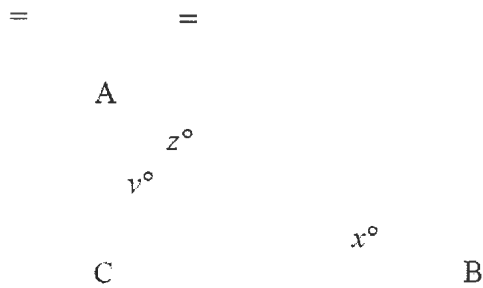
Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form.

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H

11. Let k be the smallest integer and let w be the largest integer such that both k and w are values for x that satisfy the inequality $|2x - 1| < 15$. Find the value of $(3k + 2w)$.

12. In rhombus $ABCD$, $\angle DAB = 60^\circ$. A circle passes through vertices A , B , and D . The center of the circle is P . Find $\angle BPC$.



15. A circle has an equation of $(x - 8)^2 + y^2 + 10y = -13$. Find the length of the radius of the circle. Give your

answer in the form $a\sqrt{b}$ where both a and b are integers and b is prime.

16. In the figure, $\angle A = 40^\circ$, $\angle B = 70^\circ$, and $\angle C = 110^\circ$. Find $\angle D$.

Name:

Team Code:

2020 John O'Bryan Mathematical Competition

Junior/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form.

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Name: _____

Team Code: _____

2020 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blank on the answer sheet and in simplest form.

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2. Assume x, y, z are integers with $x \geq y > 0 > z$. Consider the equation

$$x^2 + y^2 + z^2 = x^3 + y^3 + z^3.$$

(a) Find a solution where $|x| = |z|$ and $|x| + |y| + |z| < 10$.

Solution: Since $|x| = |z|$ and $x > 0 > z$, we have $z = -x$, which gives

$$2x^2 + y^2 = y^3.$$

Solving for x gives $x = \sqrt{\frac{y^3 - y^2}{2}} = y\sqrt{\frac{y-1}{2}}$. Since x is an integer, $\frac{y-1}{2}$ must be a perfect square. Since $x + y + |z| < 10$ and $x \geq y$, $\frac{y-1}{2} = 1$. Therefore, $y = 3$, which makes $x = 3$ and $z = -3$.

(b) Find a solution where $y = \frac{x}{2}$.

Solution: To construct an example, when $y = \frac{x}{2}$, we have

$$\begin{aligned} 0 &= x^3 + y^3 + z^3 - x^2 - y^2 - z^2 \\ &= \frac{9x^3}{8} + z^3 - \frac{5x^2}{4} - z^2. \end{aligned}$$

To eliminate the z terms, pick $z = -x$ so that

$$\begin{aligned} 0 &= \frac{x^3}{8} - \frac{9x^2}{4} \\ &= \frac{x^2}{4} \left(\frac{x}{2} - 9 \right), \end{aligned}$$

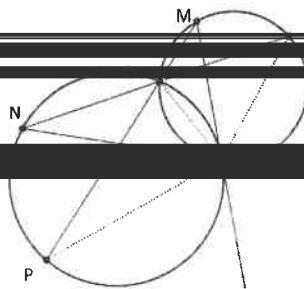
which implies $x = 18$, making $y = 9$ and $z = -18$.

(c) Show that there are infinitely many solutions to the equation.

Solution: When $z = -x$, the analysis in part (a) gives that the equation is satisfied whenever



3. In the circles below, note that: (1) the circles intersect at A and B ; (2) \overrightarrow{MA} is tangent to the circle containing B ; and (3) \overrightarrow{NA} is tangent to the circle containing P .



- (a) Prove that triangles AQN and AMP are similar.

Solution: By the Inscribed Angle Theorem, $\angle P = \angle N$ and $\angle Q = \angle M$. Combined with the angle sum formula, these imply that $\angle NAQ = \angle PAM$. Since both triangles have same angle measures, they are similar.

- (b) Prove that $\angle ABQ = \angle MAN$.

Solution: Note that angle ABQ is an exterior angle of the triangle ABN , so $\angle ABQ =$

4. Let $a \neq -1$.

(a) Calculate $\frac{a^5 + 1}{a + 1}$.

Solution: Note that $a^5 + 1 = (a + 1)(a^4 - a^3 + a^2 - a + 1)$.

(b) Assume $a + 1$ is divisible by 5. Show that $a^4 - a^3 + a^2 - a + 1$ is divisible by 5.

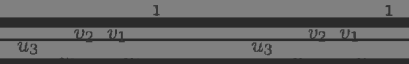
Solution: Since 5 divides $a + 1$, $a + 1 = 5m$ for some $m \in \mathbb{Z}$. Thus $a = 5m - 1$. As such

$$\begin{aligned} a^4 - a^3 + a^2 - a + 1 &= (5m - 1)^4 - (5m - 1)^3 + (5m - 1)^2 - (5m - 1) + 1 \\ &= (5k_4 + 1) - (5k_3 - 1) + (5k_2 + 1) - (5m - 1) + 1 \\ &= 5(k_4 - k_3 + k_2 - m) + 5 \\ &= \end{aligned}$$

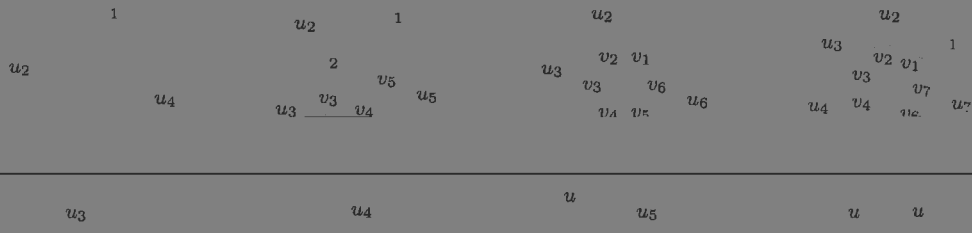
5. In discrete time steps (i.e. $t = 0, 1, 2, 3, 4, \dots$), a contagion spreads around vertices in a graph where infected vertices are solid and healthy vertices are open. If an infected vertex has only one healthy



v_3



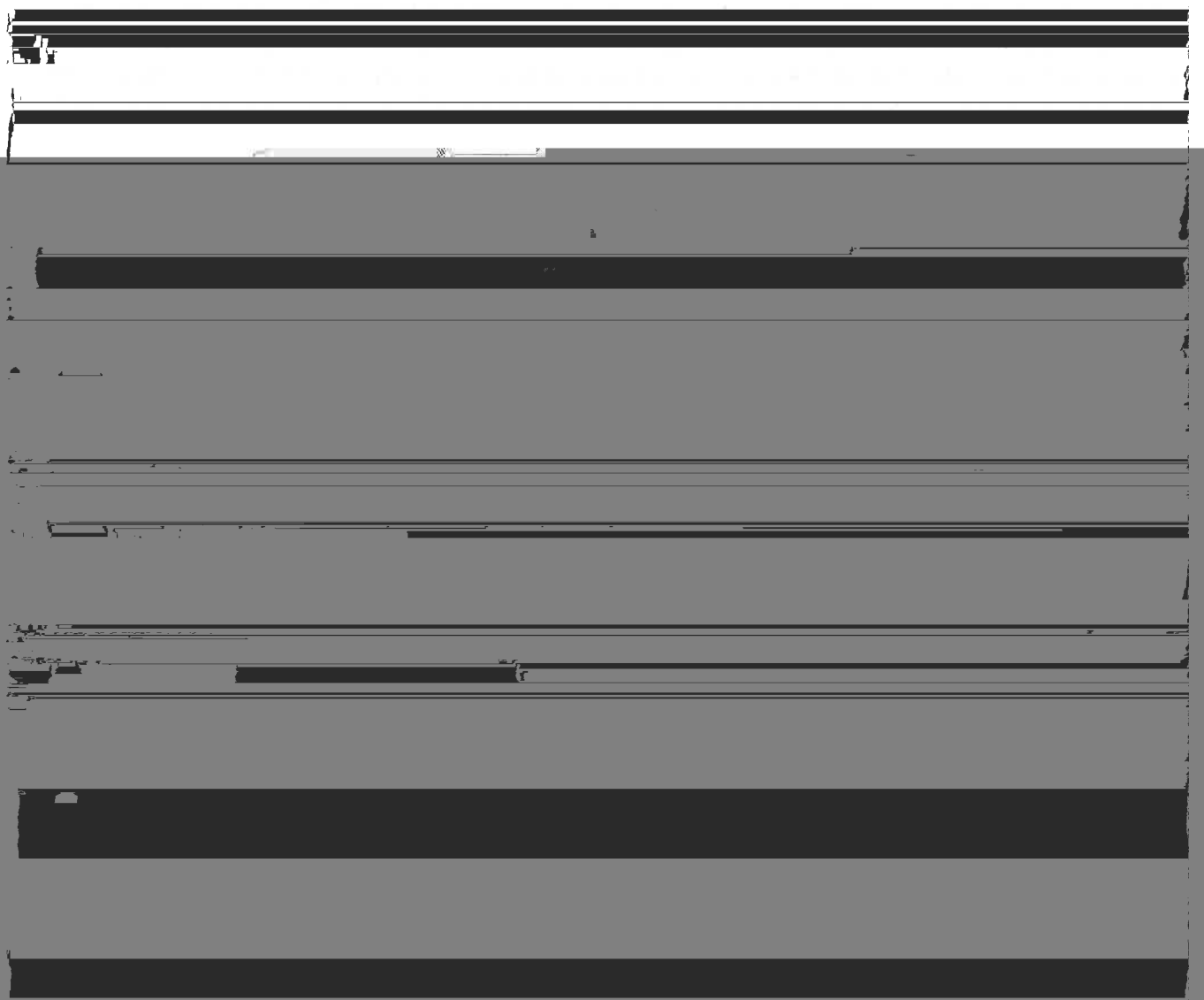
graph, C_n , has n vertices in a circle, each with a single additional neighbor; see examples of C_4 , C_5 , C_6 , and C_7 below.



(a) If u_1, u_4 are initially infected, how long does it take to completely infect C_4 ?

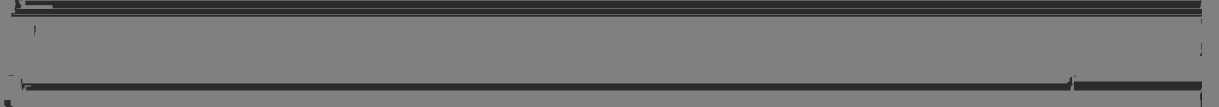
In general, C_{4m} needs $2m$ initial and $m+1$ steps; C_{4m+1} needs $2m+1$ initial and $m+2$ steps;
 C_{4m+2} needs $2m+1$ initial and $m+2$ steps and C_{4m+3} needs $2m+2$ initial and $m+2$ steps.

6. In chess, a rook (i.e. castle) can move any number of spaces vertically or horizontally. We define the *rook social distance* between squares on a chess board as the number of moves it takes a rook to move



		♞	6	5	5
1	2	♞	5	5	
		3	4		
4	3	4		4	
		3	4		

Suppose there are only 3 rooks. If they are in the same row, then the maximum rook social



Name: _____ ANSWERS _____

Team Code: _____

2020 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 1507

11. 1

2. 500 Percent optional

12. 144

3. -23

13. 16

4. 1

14. 132

5. 2

15. $2\sqrt{3}$ Must be this exact answer.

6. 101

16. 46

7. $\frac{4}{7}$ Must be this reduced fraction.

17. 50

8. 1050 (cents optional)

18. 65

9. 48

19. $2\sqrt{5}$ Must be this exact answer.

10. 2346

20. 25

